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On Efficiency and Duality for Multiobjective Programs

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For a multiobjective nonlinear program which involved inequality and equality constraints, Wolfe, Mond–Weir, and general Mond–Weir type duals are formulated and the concept of efficiency (Pareto optimum) is used to state some duality results under generalized (F, ρ)-convexity assumptions.

1. INTRODUCTION AND PRELIMINARIES

In this paper, our aim is to use the concept of efficiency (Pareto optimum) to formulate some results of duality under generalized (F, ρ)-convexity assumptions for the following class of multiobjective nonlinear program:

$$(\mathrm{VOP}) \begin{cases} \text{minimize } \ (f_1(x), f_2(x), ..., f_p(x)) \\ \text{subject to } g(x) \leqslant 0,\ h(x) = 0, \end{cases}$$

where $f_i: \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, ..., p$, $g = (g_1, g_2, ..., g_m)$, $g_j: \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, ..., m$, $h = (h_1, h_2, ..., h_q)$, $h_k: \mathbb{R}^n \to \mathbb{R}$, $k = 1, 2, ..., q$ are assumed to be differentiable.

Following Egudo [3] we consider for problem (VOP) the Wolfe vector dual and Mond–Weir vector dual. Further in the last section a general Mond–Weir vector dual is formulated and some duality results are stated. In the case $h \equiv 0$ and different assumptions of convexity (convexity, generalized convexity, or generalized ρ-convexity), Weir [9, 10] and Egudo [2, 3] have used proper efficiency [4] or efficiency to establish some duality results, where Wolfe and Mond–Weir duals are considered. Proofs of strong duality results use characterizations of proper efficiency and efficiency of Geoffrion [4] and Chankong and Haimes [1], respectively. Also we shall use the characterization of efficiency from [1, Theorem 4.1].

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LEMMA 1. *x^0 is an efficient solution for* (VOP) *if and only if x^0 solves*

$$P_k(x^0)\begin{cases}\text{minimize } f_k(x)\\ \text{subject to } f_j(x)\leqslant f_j(x^0) \text{ for all } j\neq k,\\ \qquad g(x)\leqslant 0,\ h(x)=0\end{cases}$$

for each $k=1, 2, ..., p$.

We recall the following notation. For $x, y\in\mathbb{R}^n$, by $x\leqslant y$, we mean $x_i\leqslant y_i$ for all i.

Now we consider a general type of convexity, namely generalized (F, ρ)-convexity, an extension of F-convexity defined by Hanson and Mond [5] and generalized ρ-convexity defined by Vial [7, 8].

DEFINITION 1. A functional $F: X\times X\times\mathbb{R}^n\to\mathbb{R}$ (where $X\subseteq\mathbb{R}^n$) is sublinear if for any $x, x^0\in X$,

$$F(x, x^0; a_1+a_2)\leqslant F(x, x^0; a_1)+F(x, x^0; a_2), \qquad \text{for any} \quad a_1, a_2\in\mathbb{R}^n$$

and

$$F(x, x^0; \alpha a)=\alpha F(x, x^0; a), \qquad \text{for any} \quad \alpha\in\mathbb{R}, \alpha\geqslant 0, \text{ and } a\in\mathbb{R}^n.$$

Let us consider a sublinear functional F and the function $\varphi: X\to\mathbb{R}$. We suppose that φ is with first partial derivatives at x^0, an interior point of X, and $\nabla\varphi(x^0)$ is the gradient vector of φ at x^0. Let $d(\cdot, \cdot)$ be a pseudometric on $\mathbb{R}^n$, and $\rho\in\mathbb{R}$.

DEFINITION 2. The function φ is said to be (F, ρ)-convex at x^0, if for all $x\in X$ we have

$$\varphi(x)-\varphi(x^0)\geqslant F(x, x^0; \nabla\varphi(x^0))+\rho d^2(x, x^0).$$

This function φ is said to be strongly F-convex, F-convex, or weakly F-convex at x^0, according to $\rho>0$, $\rho=0$, or $\rho<0$.

DEFINITION 3. The function φ is (F, ρ)-quasiconvex at x^0 if for all $x\in X$ such that $\varphi(x)\leqslant\varphi(x^0)$ we have

$$F(x, x^0; \nabla\varphi(x^0))\leqslant -\rho\, d^2(x, x^0).$$

We say that φ is strongly F-quasiconvex, F-quasiconvex, or weakly F-quasiconvex at x^0 according to $\rho>0$, $\rho=0$, or $\rho<0$.

DEFINITION 4. The function φ is (F, ρ)-pseudoconvex at x^0 if for all $x \in X$ such that $F(x, x^0; \nabla\varphi(x^0)) \geqslant -\rho\, d^2(x, x^0)$ it results $\varphi(x) \geqslant \varphi(x^0)$.

This function φ is strongly F-pseudoconvex, F-pseudoconvex, or weakly F-pseudoconvex according to $\rho > 0$, $\rho = 0$, or $\rho < 0$.

DEFINITION 5. We say that φ is strictly (F, ρ)-pseudoconvex at x^0 if for all $x \in X$, $x \neq x^0$ such that $F(x, x^0; \nabla\varphi(x^0)) \geqslant -\rho\, d^2(x, x^0)$ it results $\varphi(x) > \varphi(x^0)$, or equivalently, if $\varphi(x) \leqslant \varphi(x^0)$ we have $F(x, x^0; \nabla\varphi(x^0)) < -\rho\, d^2(x, x^0)$.

2. WOLFE VECTOR DUALITY

In this section we consider weak and strong duality relations between (VOP) and the following Wolfe vector dual [11]:

$$\text{(WVD)} \begin{cases} \text{maximize } (f_1(u) + y^T g(u) + z^T h(u), ..., f_p(u) + y^T g(u) + z^T h(u)) \\ \text{subject to } [\nabla f(u)]^T \alpha + [\nabla g(u)]^T y + [\nabla h(u)]^T z = 0 \\ \qquad \alpha^T e = 1 \\ \qquad y \geqslant 0, \alpha \geqslant 0 \\ \qquad \alpha \in \mathbb{R}^p,\ y \in \mathbb{R}^m,\ z \in \mathbb{R}^q, \end{cases} \tag{1}$$

where $e = (1, 1, ..., 1)^T \in \mathbb{R}^p$.

In the first theorem a scalarization of the objective and constraints functions is considered.

THEOREM 1 (Weak Duality). *Assume that for all feasible x for* (VOP) *and all feasible (u, α, y, z) for* (WVD),

(a) *$\alpha^T f(\cdot) + y^T g(\cdot) + z^T h(\cdot)$ is F-convex at u. Further, if either*

(b) *$\alpha_i > 0$ for all $i \in P = \{1, 2, ..., p\}$, or*

(c) *$\alpha^T f(\cdot) + y^T g(\cdot) + z^T h(\cdot)$ is strictly F-convex at u*

then the following cannot hold

$$f_j(x) \leqslant f_j(u) + y^T g(u) + z^T h(u), \qquad \textit{for all} \quad j \in P \tag{2}$$

and

$$f_i(x) < f_i(u) + y^T g(u) + z^T h(u) \qquad \textit{for some} \quad i \in P. \tag{3}$$

Proof. Suppose to the contrary that (2) and (3) hold. Since x is feasible for (VOP) and $y \geqslant 0$, (2) and (3) imply

$$f_j(x) + y^T g(x) + z^T h(x) \leqslant f_j(u) + y^T g(u) + z^T h(u) \qquad \text{for all } j, \tag{4}$$

and

$$f_i(x) + y^T g(x) + z^T h(x) < f_i(u) + y^T g(u) + z^T h(u) \qquad \text{for some } i, \tag{5}$$

respectively. Then in view of hypothesis (b) and $\alpha^T e = 1$, we obtain

$$\alpha^T f(x) + y^T g(x) + z^T h(x) < \alpha^T f(u) + y^T g(u) + z^T h(u). \tag{6}$$

According to (a) and (6) we have

$$F(x, u; [\nabla f(u)]^T \alpha + [\nabla g(u)]^T y + [\nabla h(u)]^T z) < 0 \tag{7}$$

which contradicts (1) because $F(x, u; 0) = 0$.

When the hypothesis (c) holds, since $\alpha_i \geqslant 0$, $i \in P$, and $\alpha^T e = 1$, (4) and (5) imply

$$\alpha^T f(x) + y^T g(x) + z^T h(x) \leqslant \alpha^T f(u) + y^T g(u) + z^T h(u)$$

and then again we have (7). Also we obtain a contradiction, and the proof is complete.

The next theorem requires various levels of convexity on the component of the functions involved.

THEOREM 1′ (Weak Duality). *Assume that for all feasible x for* (VOP) *and all feasible* (u, α, y, z) *for* (WVD),

(a) f_i, $i \in P$; g_j, $j = 1, 2, ..., m$; h_k, $-h_k$, $k = 1, 2, ..., q$ *are F-convex.*

If also either (b) *or* (c) *from Theorem* 1 *is satisfied, then* (2) *and* (3) *cannot hold.*

Proof. Since f_i, $i \in P$, are F-convex we have

$$f_i(x) - f_i(u) \geqslant F(x, u; \nabla f_i(u)), \qquad i \in P. \tag{8}$$

By (8), $\alpha^T e = 1$ and from the sublinearity of F we obtain

$$\alpha^T f(x) - \alpha^T f(u) \geqslant F(x, u; [\nabla f(u)]^T \alpha). \tag{9}$$

Because $y \geqslant 0$ and g_j, $j = 1, 2, ..., m$, are F-convex and F is sublinear we have

$$y^T g(x) - y^T g(u) \geqslant F(x, u; [\nabla g(u)]^T y). \tag{10}$$

Let $z = z_1 - z_2$, $z_1, z_2 \geqslant 0$. As above, we can write

$$z_1^T h(x) - z_1^T h(u) \geqslant F(x, u; [\nabla h(u)]^T z_1)$$
$$-z_2^T h(x) + z_2^T h(u) \geqslant F(x, u; -[\nabla h(u)]^T z_2)$$

and then we obtain

$$z^T h(x) - z^T h(u) \geqslant F(x, u; [\nabla h(u)]^T z). \tag{11}$$

From (9), (10), (11), and the sublinearity of F it results

$$\alpha^T f(x) + y^T g(x) + z^T h(x) - (\alpha^T f(u) + y^T g(u) + z^T h(u))$$
$$\geqslant F(x, u; [\nabla f(u)]^T \alpha + [\nabla g(u)]^T y + [\nabla h(u)]^T z)$$

and now the proof is similar to that of Theorem 1.

Now a weak duality result between (VOP) and (WVD) under (F, ρ)-convexity is considered. Here this assumption is used only for a scalarization of f, g, and h.

THEOREM 2 (Weak Duality). *Assume that for all feasible x for* (VOP) *and all feasible* (u, α, y, z) *for* (WVD),

(a) $\alpha^T f(\cdot) + y^T g(\cdot) + z^T h(\cdot)$ *is (F, ρ)-convex at u. Further, if either*

(b) $\alpha_i > 0$, *for all $i \in P$ and $\rho \geqslant 0$ or*

(c) $\rho > 0$, *and $d(\cdot, \cdot)$ is a metric on $\mathbb{R}^n$*,

then (2) *and* (3) *cannot hold.*

Proof. We also suppose contrary to the result that (2) and (3) hold. Because x and (u, α, y, z) are feasible solutions for (VOP) and (WVD), respectively, in the case (b) we find

$$\alpha^T f(x) + y^T g(x) + z^T h(x) < \alpha^T f(u) + y^T g(u) + z^T h(u). \tag{12}$$

Now from (12) and (a) we obtain

$$F(x, u; [\nabla f(u)]^T \alpha + [\nabla g(u)]^T y + [\nabla h(u)]^T z) < -\rho\, d^2(x, u) \tag{13}$$

and then from (1) and the sublinearity of F, this implies $\rho\, d^2(x, u) < 0$ which is a contradiction with the fact that $\rho \geqslant 0$.

When we have (c) from (2), (3) and $\alpha \geqslant 0$, we obtain

$$\alpha^T f(x) + y^T g(x) + z^T h(x) \leqslant \alpha^T f(u) + y^T g(u) + z^T h(u)$$

and then by (a) we find that

$$F(x, u; [\nabla f(x)]^T\alpha + [\nabla g(u)]^T y + [\nabla h(u)]^T z) \leqslant -\rho\, d^2(x, u).$$

But this violates (1). Hence the proof is complete.

THEOREM 2′ (Weak Duality). *Assume that for all feasible x for* (VOP) *and all feasible* (u, α, y, z) *for* (WVD),

(a_1) *f_i is (F, ρ_{1i})-convex, $i \in P$;*

(a_2) *g_j is (F, ρ_{2j})-convex, $j = 1, 2, ..., m$;*

(a_3) *h_k is (F, ρ_{3k})-convex, $-h_k$ is (F, ρ_{4k})-convex with $\rho_{3k} + \rho_{4k} \geqslant 0$, $1 \leqslant k \leqslant q$. Also if either*

(b) *$\alpha_i > 0$, for all $i \in P$ and $\sum_{i=1}^{p} \rho_{1i}\alpha_i + \sum_{j=1}^{m} \rho_{2j} y_j + \sum_{k=1}^{q} \rho_{3k} z_k \geqslant 0$*

or

(c) *$\sum_{i=1}^{p} \rho_{1i}\alpha_i + \sum_{j=1}^{m} \rho_{2j} y_j + \sum_{k=1}^{q} \rho_{3k} z_k > 0$ and $d(\cdot, \cdot)$ is a metric on $\mathbb{R}^n$, then* (2) *and* (3) *cannot hold.*

Proof. We proceed as in the proof of Theorems 1′ and 2.

COROLLARY 1. *Let $(u^0, \alpha^0, y^0, z^0)$ be a feasible solution for* (WVD) *such that $y^{0T}g(u^0) = 0$ and assume that u^0 is feasible for* (VOP). *If weak duality* (*any of Theorems* 1, 1′, 2, 2′) *holds between* (VOP) *and* (WVD), *then u^0 is efficient for* (VOP) *and $(u^0, \alpha^0, y^0, z^0)$ is efficient for* (WVD).

Proof. This follows on the lines of Egudo [3, Corollary 1] along with one of Theorems 1, 1′, 2, 2′ from above and equality constraints.

Also we can formulate a strong duality result:

THEOREM 3 (Strong Duality). *Let x^0 be a feasible solution for* (VOP) *and assume that*

(i) *x^0 is an efficient solution*

(ii) *for at least one i, $i \in P$, x^0 satisfies a constraint qualification for problem $P_i(x^0)$.*

Then there exist $\alpha^0 \in \mathbb{R}^p$, $y^0 \in \mathbb{R}^m$, $z^0 \in \mathbb{R}^q$ such that $(x^0, \alpha^0, y^0, z^0)$ is feasible for (WVD) *and $y^{0T}g(x^0) = 0$.*

Further if also weak duality (*any of Theorems* 1, 1′, 2, 2′) *holds between* (VOP) *and* (WVD) *then $(x^0, \alpha^0, y^0, z^0)$ is efficient for* (WVD).

Proof. This follows on the lines of Egudo [3, Theorem 3] along with Corollary 1 of above.

3. Mond–Weir Vector Duality

Here, we establish various duality theorems for the Mond–Weir dual [6] of problem (VOP) defined in Egudo [3].

$$
\text{(DVOP)}\begin{cases}\text{maximize } (f_1(u), f_2(u), ..., f_p(u))\\ \text{subject to } [\nabla f(u)]^T\alpha + [\nabla g(u)]^T y + [\nabla h(u)]^T z = 0\\ \qquad y^T g(u) \geqslant 0\\ \qquad z^T h(u) = 0\\ \qquad \alpha^T e = 1\\ \qquad y \geqslant 0,\ \alpha \geqslant 0.\end{cases}
$$

The weak duality results are given under conditions of generalized (F, ρ)-convexity and (F, ρ)-convexity, where only assumption (a) from the next theorem involves a scalarization of g and h.

THEOREM 4 (Weak Duality). *Assume that for all feasible s for* (VOP) *and all feasible* (u, α, y, z) *for* (DVOP),

(a) $y^T g(\cdot) + z^T h(\cdot)$ *is* (F, ρ)*-quasiconvex at* u*, and if also any of the following holds*

(b) $\alpha_i > 0$ *for all* $i \in P$*, and* f_i *is* (F, ρ_{1i})*-pseudoconvex at* u *for any* $i \in P$*, with* $\rho + \sum_{i=1}^{p} \rho_{1i}\alpha_i \geqslant 0$;

(c) $\alpha_i > 0$ *for all* $i \in P$ *and* $\alpha^T f(\cdot)$ *is* (F, ρ')*-pseudoconvex at* u*, with* $\rho + \rho' \geqslant 0$;

(d) $\alpha^T f(\cdot)$ *is strictly* (F, ρ')*-pseudoconvex at* u*, with* $\rho + \rho' > 0$,

then the following cannot hold

$$f_j(x) \leqslant f_j(u) \qquad \text{for all} \quad j \in P \tag{14}$$

and

$$f_i(x) < f_i(u) \qquad \text{for some} \quad i \in P. \tag{15}$$

Proof. Let x be an arbitrary feasible solution of (VOP) and (u, α, y, z) be an arbitrary feasible solution of (DVOP). Then in view of $y \geqslant 0$ we have that $y^T g(x) \leqslant y^T g(u)$, $z^T h(x) = z^T h(u)$. Hence,

$$y^T g(x) + z^T h(x) \leqslant y^T g(u) + z^T h(u),$$

and since $y^T g(\cdot) + z^T h(\cdot)$ is (F, ρ)-quasiconvex at u, this implies

$$F(x, u; [\nabla g(u)]^T y + [\nabla h(u)]^T z) \leqslant -\rho\, d^2(x, u). \tag{16}$$

From (16), feasibility of (u, α, y, z), and sublinearity of F we obtain

$$F(x, u; [\nabla f(u)]^T \alpha) \geqslant \rho\, d^2(x, u). \tag{17}$$

On the other hand, suppose contrary to the result of the theorem that (14) and (15) hold. If we have the hypothesis (b), then (14), (15), and (F, ρ_{1i})-pseudoconvexity of f_i, $i \in P$, imply

$$F(x, u; \nabla f_j(u)) \leqslant -\rho_{ij}\, d^2(x, u), \qquad \text{for all} \quad j \in P \tag{18}$$

and

$$F(x, u; \nabla f_i(u)) < -\rho_{1i}\, d^2(x, u) \qquad \text{for some} \quad i \in P. \tag{19}$$

Because $\alpha_i > 0$ for all $i \in P$, from (18), (19), and the sublinearity of F we have

$$F(x, u; [\nabla f(u)]^T \alpha) < -\left(\sum_{i=1}^{p} \rho_{1i} \alpha_i \right) d^2(x, u) \tag{20}$$

which is in contradiction to (17) because $\rho + \sum_{i=1}^{p} \rho_{1i} \alpha_i \geqslant 0$.

When the hypothesis (c) holds, from (14) and (15) we obtain $\alpha^T f(x) < \alpha^T f(u)$ and then we have a contradiction to (17). In the last case, if the hypothesis (d) holds, from (14) and (15) we have $\alpha^T f(x) \leqslant \alpha^T f(u)$ and then strictly (F, ρ')-pseudoconvexity of $\alpha^T f(\cdot)$ at u implies again a contradiction to (17). The proof is complete.

THEOREM 5 (Weak Duality). *Assume that for all x for* (VOP) *and all feasible* (u, α, y, z) *for* (DVOP),

(a_1) f_i *is* (F, ρ_{1i})*-convex*, $i \in P$;

(a_2) g_j *is* (F, ρ_{2j})*-convex*, $j = 1, 2, ..., m$;

(a_3) h_k *is* (F, ρ_{3k})*-convex*, $-h_k$ *is* (F, ρ_{4k})*-convex, with* $\rho_{3k} + \rho_{4k} \geqslant 0$ *for all* $k = 1, 2, ..., q$.

Further if also any of the following holds

(b) $\alpha_i > 0$ *for all* $i \in P$ *and* $\sum_{i=1}^{p} \alpha_i \rho_{1i} + \sum_{j=1}^{m} y_j \rho_{2j} + \sum_{k=1}^{q} z_k \rho_{3k} \geqslant 0$;

(c) $\sum_{i=1}^{p} \alpha_i \rho_{1i} + \sum_{j=1}^{m} y_j \rho_{2j} + \sum_{k=1}^{q} z_k \rho_{3k} > 0$, *and* $d(\ ,\ )$ *is a metric*,

then (14) *and* (15) *cannot hold.*

Proof. Suppose to the contrary that (14) and (15) hold. Then from (14), (15), and (a_1) we have

$$F(x, u; \nabla f_i(u)) + \rho_{1i}\, d^2(x, u) \leqslant 0, \qquad \text{for all} \quad i \in P \tag{21}$$

and with a strict inequality for some i. From (a_2) we have

$$g_j(x) - g_j(u) \geqslant F(x, u; \nabla g_j(u)) + \rho_{2j}\, d^2(x, u) \qquad \text{for all } j,$$

and because $y \geqslant 0$, $y^T g(x) \leqslant 0$, $y^T g(u) \geqslant 0$, and F is sublinear we obtain

$$F(x, u; [\nabla g(u)]^T y) + \left(\sum_{j=1}^{m} \rho_{2j} y_j \right) d^2(x, u) \leqslant 0. \tag{22}$$

Also from (a_3) we have

$$z^{1T} h(x) - z^{1T} h(u) \geqslant F(x, u; [\nabla h(u)]^T z^1) + \left(\sum_{k=1}^{q} \rho_{3k} z_{1k} \right) d^2(x, u) \tag{23}$$

$$-z^{2T} h(x) + z^{2T} h(u) \geqslant F(x, u; -[\nabla h(u)]^T z^2) + \left(\sum_{k=1}^{q} \rho_{4k} z_{2k} \right) d^2(x, u), \tag{24}$$

where $z = z^1 - z^2$, $z^1 \geqslant 0$, $z^2 \geqslant 0$, $z^1 = (z_{11}, z_{12}, ..., z_{1q})^T$, $z^2 = (z_{21}, ..., z_{2q})^T$. Thus, adding (23), (24), and applying sublinearity of F and again the hypothesis (a_3) we obtain

$$z^T h(x) - z^T h(u) \geqslant F(x, u; [\nabla h(u)]^T z) + \left(\sum_{k=1}^{q} \rho_{3k} z_k \right) d^2(x, u).$$

This relation together with $h(x) = 0$, $z^T h(u) = 0$, yields

$$F(x, u; [\nabla h(u)]^T z) + \left(\sum_{k=1}^{q} \rho_{3k} z_k \right) d^2(x, u) \leqslant 0. \tag{25}$$

From (22), (25), sublinearity of F, and (1) we have

$$F(x, u; [\nabla f(u)]^T \alpha) \geqslant \left(\sum_{j=1}^{m} \rho_{2j} y_j + \sum_{k=1}^{q} \rho_{3k} z_k \right) d^2(x, u). \tag{26}$$

Now, if the hypothesis (b) holds, from (21) and sublinearity of F we obtain

$$F(x, u; [\nabla f(u)]^T \alpha) < \left(\sum_{j=1}^{m} \rho_{2j} y_j + \sum_{k=1}^{q} \rho_{3k} z_k \right) d^2(x, u) \tag{27}$$

contradicting (26).

When (c) holds, the inequality (21) and sublinearity of F also imply (27). Thus the proof is complete.

The above proof suggests the following modification of Theorem 5.

THEOREM 5′. *Assume that for all feasible x for* (VOP) *and all feasible* (u, α, y, z) *for* (DVOP),

(a) $\alpha^T f(\cdot) + y^T g(\cdot) + z^T h(\cdot)$ *is* (F, ρ)*-convex, and either*

(b) $\alpha_i > 0$ *for all* $i \in P$ *and* $\rho \geqslant 0$, *or*

(c) $\rho > 0$, *and* $d(\cdot, \cdot)$ *is a metric,*

then (14) *and* (15) *cannot hold.*

COROLLARY 2. *Assume weak duality* (*any of Theorems* 4, 5, 5′) *holds between* (VOP) *and* (DVOP). *If* $(u^0, \alpha^0, y^0, z^0)$ *is feasible for* (DVOP) *such that* u^0 *is feasible for* (VOP), *then* u^0 *is efficient for* (VOP) *and* $(u^0, \alpha^0, y^0, z^0)$ *is efficient for* (DVOP).

Proof. It follows on the lines of Egudo [3, Corollary 2] along with one of Theorems 4, 5, 5′ above.

Now, following Egudo [3] and Corollary 2 above we obtain a strong duality result.

THEOREM 6 (Strong Duality). *Let* x^0 *be a feasible solution for* (VOP) *and assume that*

(a) x^0 *is efficient*

(b) x^0 *satisfies a constraint qualification for* $P_i(x^0)$ *for at least one* $i \in P$.

Then there exist $\alpha^0 \in \mathbb{R}^p$, $y^0 \in \mathbb{R}^m$, $z^0 \in \mathbb{R}^q$ *such that* $(x^0, \alpha^0, y^0, z^0)$ *is feasible for* (DVOP).

Further, if also weak duality (*any of Theorems* 4, 5, 5′) *holds between* (VOP) *and* (DVOP) *then* $(x^0, \alpha^0, y^0, z^0)$ *is efficient for* (DVOP).

4. Generalized Mond–Weir Duality

We shall continue our discussion of duality for (VOP) in the present section by introducing a general dual problem for (VOP) and proving weak and strong duality theorems under generalized (F, ρ)-convexity conditions.

Consider the following problem

$$
\text{(DMW)} \begin{cases} \text{maximize } (f_1(u) + y_{J_0}^T g_{J_0}(u) + z_{k_0}^T h_{k_0}(u), \ldots, f_p(u) \\ \qquad + y_{J_0}^T g_{J_0}(u) + z_{k_0}^T h_{k_0}(u)) \\ \text{subject to } [\nabla f(u)]^T \alpha + [\nabla g(u)]^T y + [\nabla h(u)]^T z = 0 \\ \qquad y_{J_t}^T g_{J_t}(u) + z_{k_t}^T h_{k_t}(u) \geqslant 0, \ 1 \leqslant t \leqslant v \\ \qquad \alpha^T e = 1 \\ \qquad y \geqslant 0, \ \alpha \geqslant 0, \end{cases} \tag{28}
$$

with notation $y_{J_t}^T g_{J_t}(u)=\sum_{j\in J_t} y_j g_j(u)$, $z_{k_t}^T h_{k_t}(u)=\sum_{k\in K_t} z_k h_k(u)$, where J_t, K_t, $0\leqslant t\leqslant v$ are partitions of the sets $\{1, 2, ..., m\}$, $\{1, 2, ..., q\}$, respectively, and $v=\max\{v_1, v_2\}$ where v_1, v_2 is the number of partitions of $\{1, 2, ..., m\}$ and $\{1, 2, ..., q\}$, respectively, and $J_t=\varnothing$ or $K_t=\varnothing$ for $t>\min\{v_1, v_2\}$.

THEOREM 7 (Weak Duality). *Assume that for all feasible x for* (VOP) *and all feasible* (u, α, y, z) *for* (DMW),

(a) $y_{J_t}^T g_{J_t}(\cdot)+z_{k_t}^T h_{k_t}(\cdot)$ *is (F, ρ_{1t})-quasiconvex at u for any t, $1\leqslant t\leqslant v$, and also any of the following holds*

(b) $\alpha_i>0$ *for any $i\in P$ and* $f_i(\cdot)+y_{J_0}^T g_{J_0}(\cdot)+z_{k_0}^T h_{k_0}(\cdot)$ *is (F, ρ_{2i})-pseudoconvex at u, $i\in P$, with* $\sum_{t=1}^{v} \rho_{1t}+\sum_{i=1}^{p} \alpha_i\rho_{2t}\geqslant 0$;

(c) $\alpha_i>0$ *for all $i\in P$ and* $\alpha^T f(\cdot)+y_{J_0}^T g_{J_0}(\cdot)+z_{k_0}^T h_{k_0}(\cdot)$ *is (F, ρ)-pseudoconvex at u, with* $\rho+\sum_{t=1}^{v} \rho_{1t}\geqslant 0$;

(d) $\alpha^T f(\cdot)+y_{J_0}^T g_{J_0}(\cdot)+z_{k_0}^T h_{k_0}(\cdot)$ *is strictly (F, ρ)-pseudoconvex at u, with* $\rho+\sum_{t=1}^{v} \rho_{1t}\geqslant 0$.

Then the following cannot hold

$$f_j(x)\leqslant f_j(u)+y_{J_0}^T g_{J_0}(u)+z_{k_0}^T h_{k_0}(u) \qquad \textit{for all} \quad j\in P \tag{29}$$

and

$$f_i(x)<f_i(u)+y_{J_0}^T g_{J_0}(u)+z_{k_0}^T h_{k_0}(u) \qquad \textit{for some} \quad i\in P. \tag{30}$$

Proof. Let x be an arbitrary feasible solution of (VOP) and (u, α, y, z) be an arbitrary feasible solution of (DMW). Then we have

$$y_{J_t}^T g_{J_t}(x)+z_{k_t}^T h_{k_t}(x)\leqslant y_{J_t}^T g_{j_t}(u)+z_{k_t}^T h_{k_t}(u) \tag{31}$$

for all $t=1, 2, ..., v$. In view of (31) and the hypothesis (a), we obtain

$$F(x, u; [\nabla g_{J_t}(u)]^T y_{J_t}+[\nabla h_{k_t}(u)]^T z_{k_t})\leqslant -\rho_{1t}\, d^2(x, u). \tag{32}$$

Relation (32) and sublinearity of F yields

$$F\left(x, u; \sum_{t=1}^{v} ([\nabla g_{J_t}(u)]^T y_{J_t}+[\nabla h_{k_t}(u)]^T z_{k_t})\right) \leqslant -\left(\sum_{t=1}^{v} \rho_{1t}\right) d^2(x, u). \tag{33}$$

Also from sublinearity of F, (28), (33), and because $F(x, u; 0) = 0$, we have

$$F(x, u; [\nabla f(u)]^T \alpha + [\nabla g_{J_0}(u)]^T y_{J_0} + [\nabla h_{k_0}(u)]^T z_{k_0}) \geqslant \left(\sum_{t=1}^{v} \rho_{1t} \right) d^2(x, u). \tag{34}$$

Now suppose contrary to the result of the theorem that (29) and (30) hold.

If $\alpha_i > 0$ for all $i \in P$ then (29), (30), (F, ρ_{2i})-pseudoconvexity of f_i, $i \in P$, and sublinearity of F imply

$$F(x, u; [\nabla f(u)]^T \alpha + [\nabla g_{J_0}(u)]^T y_{J_0} + [\nabla h_{k_0}(u)]^T z_{k_0}) < -\left(\sum_{i=1}^{p} \alpha_i \rho_{2i} \right) d^2(x, u). \tag{35}$$

By using (34), (35), and $\sum_{t=1}^{v} \rho_{1t} + \sum_{i=1}^{p} \alpha_i \rho_{2i} \geqslant 0$ we obtain a contradiction.

By hypothesis (c), we obtain

$$\alpha^T f(x) + y_{J_0}^T g_{J_0}(x) + z_{k_0}^T h_{k_0}(x) < \alpha^T f(u) + y_{J_0}^T g_{J_0}(u) + z_{k_0}^T h_{k_0}(u)$$

and then

$$F(x, u; [\nabla f(u)]^T \alpha + [\nabla g_{J_0}(u)]^T y_{J_0} + [\nabla h_{k_0}(u)]^T z_{k_0}) < -\rho\, d^2(x, u). \tag{36}$$

But (34), (36), and relation $\rho + \sum_{t=1}^{v} \rho_{1t} \geqslant 0$ imply again a contradiction.

When we have (d), from (29), (30), and feasibility of x and (u, α, y, z) we obtain

$$\alpha^T f(x) + y_{J_0}^T g_{J_0}(x) + z_{k_0}^T h_{k_0}(x) \leqslant \alpha^T f(u) + y_{J_0}^T g_{J_0}(u) + z_{k_0}^T h_{k_0}(u) \tag{37}$$

and then by strict (F, ρ)-pseudoconvexity of $\alpha^T f(\cdot) + y_{J_0}^T g_{J_0}^{(\cdot)} + z_{k_0}^T h_{k_0}^{(\cdot)}$ at u, (37) implies (36). Also we obtain a contradiction with (34) because $\rho + \sum_{t=1}^{v} \rho_{1t} \geqslant 0$. Thus the proof is complete.

Remarks. (i) Also for (VOP) and (DMW) we have conclusions from Corollary 2 and Theorem 6.

(ii) For this section we have a number of situations which can be obtained by appropriate choices of the partitioning sets J_t, K_t, $0 \leqslant t \leqslant v$. These situations can constitute extreme cases with respect to generalized (F, ρ)-convexity assumptions or situations lying between extremes. We have not restated these situations.

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